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The least number, of n-periodic points of a self-map of a solvmanifold, can be realised by a smooth map

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ABSTRACT

Shub and Sullivan noticed that there is a large gap between the least number of periodic points, taken over the family of continuous maps homotopic to a given self-map f and the analogous number taken over the family of smooth maps homotopic to f. In this paper we show that in the case of self-maps of solvmanifolds the minimal numbers of n-periodic points, in both categories, are equal.

1. Preliminaries

Let $f: M \to M$ be a self-map of a compact manifold. In the Nielsen periodic point theory two numbers play an important role

$$MF_n(f) = \min_{g \sim f} \#\mathrm{Fix}(g^n)$$

where g runs through the family of all continuous maps homotopic to f, and

$$MF_n^{diff}(f) = \min_{h \sim f} \#\mathrm{Fix}(h^n)$$

where h runs through the family of all smooth maps homotopic to f. By a smooth map we understand a C^1 map.

Are the numbers equal? The inequality $MF_n(f) \leqslant MF_n^{diff}(f)$ is obvious. The opposite inequality holds for $n = 1$ i.e. for fixed points [10]. However the remark of Shub and Sullivan [14] suggests that, in the case of periodic points, there is a large gap between the two numbers and the inequality is sharp.

Tori, nilmanifolds and solvmanifolds turned out to be the most convenient class of non-simply connected manifolds in the Nielsen theory. There are some nice formulae for the least number of periodic points for self-maps of these spaces [4,5]. In this paper we show that for a self-map f (of a torus, nilmanifold or solvmanifold) the numbers $MF_n(f)$ and $MF_n^{diff}(f)$ are equal. In other words in each homotopy class there is a smooth map g realising the least number of n-periodic points.

We start by recalling basic notion of the Nielsen periodic point theory. See [9,8].

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We consider a self-map of a compact polyhedron $f:X\to X$ and its fixed point set $\mathrm{Fix}(f)=\{x\in X;\ f(x)=x\}$. The *Nielsen relation*:

$x\sim y$ *iff there is a path* $\omega:[0,1]\to X$ *satisfying* $\omega(0)=x$, $\omega(1)=y$ *and* $f\omega$ *is fixed end point homotopic to* ω

splits $\mathrm{Fix}(f)$ into disjoint Nielsen classes. A Nielsen class A is called *essential* iff its fixed point index $\mathrm{ind}(f;A)\neq 0$. Let $\mathcal{N}(f)$ denote the set of Nielsen classes of the map f. The number of essential Nielsen classes in $\mathrm{Fix}(f)$ is called *Nielsen number* and is denoted by $N(f)$. This number has two basic properties: $N(f)$ is a homotopy invariant and $\#\mathrm{Fix}(g)\geqslant N(f)$ for each g homotopic to f. Thus $N(f)$ is a lower bound of the number of fixed points in the homotopy class. This bound is the best one for compact manifolds of dimension $\neq 2$: by Wecken Theorem each self-map f of such manifold is homotopic to a continuous map g having exactly $N(f)$ fixed points [15]. This theorem also holds for a large class of polyhedra although no surface with negative Euler characteristic belongs to this class [11,13].

In [10] Boju Jiang showed that Wecken Theorem holds also in the smooth category; for any self-map $f:M\to M$, of a smooth compact manifold of dimension $\neq 2$, the map realising $N(f)$ may be chosen smooth.

The Nielsen theory can be generalised to periodic points. Boju Jiang introduced (Chapter 3 in [9]) a Nielsen-type homotopy invariant $NF_n(f)$ being a lower bound of the number of n-periodic points: for each g homotopic to f

$$\mathrm{Fix}\big(g^n\big)\geqslant NF_n(f).$$

The definition of $NF_n(f)$ given in [9] is a bit complicated and here we will not need it in full generality, hence we omit it. Instead we will only use its main properties given below. Later we will use a simpler formula for $NF_n(f)$ valid in the case of nilmanifolds and solvmanifolds. It was also shown that $NF_n(f)$ is the best lower bound (Wecken-type Theorem for periodic points): each self-map $f:M\to M$ of a compact manifold of dimension $\neq 2$ is homotopic to a continuous map g satisfying $\mathrm{Fix}(g^n)=NF_n(f)$ [7]. In other words $MF_n(f)=NF_n(f)$.

Since the paper of Shub and Sullivan [14] it became clear that the differentiability assumption implies more periodic points. In fact for any self-map of a compact smooth manifold $f:M\to M$ there is defined a homotopy invariant $NJD_n(f)$ satisfying $\mathrm{Fix}(g^n)\geqslant NJD_n(f)$ for each smooth map homotopic to f [3]. This invariant satisfies $MF_n^{diff}(f)=NJD_n(f)$ if $\dim M\geqslant 3$. As above we may skip the details of this definition. We recall only that the evident inequality $NJD_n(f)\geqslant NF_n(f)$ can be sharp. For example for a map $f:S^3\to S^3$, of degree $|d|\geqslant 1$ and odd n,

$$NJD_n(f)=\mu(n)-1 \text{ or } \mu(n)$$

holds, where $\mu(n)$ denotes the number of divisors of n [2]. On the other hand $NF_n(f)=1$, since S^3 is simply-connected. More generally if the growth of the Lefschetz numbers of iterations $L(f^n)$ is faster than linear, then the above inequality is usually sharp.

In this paper we prove that in the case of self-maps of a compact nilmanifold or solvmanifold the equality $NJD_n(f)=NF_n(f)$ holds. We will show that each self-map $f:M\to M$ of a nilmanifold or a solvmanifold is homotopic to a smooth map g having exactly $NF_n(f)$ n-periodic points. Then the inequalities $NF_n(f)\leqslant NJD_n(f)\leqslant \#\mathrm{Fix}(g^n)=NF_n(g)$ imply $NF_n(f)=NJD_n(f)$.

Definition 1.1. A smooth compact manifold M is called *smoothly Wecken* if for any self-map $f:M\to M$ and a given number $n\in\mathbb{N}$, there exists a smooth map g which is homotopic to f with $\#\mathrm{Fix}(g^n)=NF_n(f)$.

Thus the title of the paper may be reformulated as

solvmanifolds are smoothly Wecken.

2. The lower bound of the number of periodic points

We will outline some basic notation of the theory introduced by Heath and Keppelmann [4,5]. To make the approach easier we assume that the considered self-map $f:X\to X$ has a fixed point x_0 which we choose for the base point. If $\mathrm{Fix}(f)=\emptyset$ then the same construction can be done with the help of a chosen path γ from x_0 to fx_0.

We consider the action of the fundamental group $\pi_1(X;x_0)$ on itself:

$$\alpha\circ\omega=\alpha\cdot\omega\cdot(f_\#\alpha)^{-1}.$$

The orbits of this action are called *Reidemeister classes* and their set is denoted by $\mathcal{R}(f)$. One may check that there is a natural map $I:\mathcal{N}(f)\to\mathcal{R}(f)$ given as follows. Let $A\subset\mathrm{Fix}(f)$ be a Nielsen class. We choose a path ω from x_0 to $x\in A$ and we define $I(A)=[\omega\cdot(f_\#\omega)^{-1}]$. This map turns out to be an injection, hence we may identify Nielsen classes with Reidemeister classes (lying in the image of I).

There is a natural action of the group $\mathbb{Z}_k$ on $\mathrm{Fix}(f^k)$ which is given by

$$\mathrm{Fix}\big(f^k\big)\ni[x]\to[fx]\in\mathrm{Fix}\big(f^k\big).$$

This induces a permutation $\mathcal{N}(f^k) \to \mathcal{N}(f^k)$ which extends to

$$\mathcal{R}\big(f^k\big) \ni [a] \to [fa] \in \mathcal{R}\big(f^k\big).$$

The orbits of this action are called *orbits of Reidemeister classes* and their set is denoted by $\mathcal{OR}(f^k)$.

On the other hand for each $l|k$ there is the natural inclusion $\mathrm{Fix}(f^l) \subset \mathrm{Fix}(f^k)$. This inclusion defines a map of Nielsen classes $\mathcal{N}(f^l) \to \mathcal{N}(f^k)$ (which may be no longer an inclusion). The last extends to the map

$$i_{kl} : \mathcal{R}\big(f^l\big) \to \mathcal{R}\big(f^k\big)$$

given by

$$i_{kl}[a] = \big[a \cdot f^l a \cdot f^{2l} a \cdots f^{k-l} a\big].$$

It is not hard to check that i_{kl} induces the map $i_{kl} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$. We will say that a Reidemeister class $A \in \mathcal{R}(f^k)$ is *reducible* if it belongs to the image of i_{kl} for an $l|k$, $l \neq k$. The *depth* of a Reidemeister class $A \in \mathcal{R}(f^k)$ is defined as the smallest number l such that $l|k$ and A belongs to the image of i_{kl}. Similarly we define the reducible (and irreducible) orbits of Reidemeister classes. Now it is not hard to prove that:

Lemma 2.1. *([4]) Each essential and irreducible orbit of Reidemeister classes in $\mathcal{R}(f^k)$ contains an orbit of points of length k.*

The above lemma and the remark that two different irreducible classes are disjoint, give the inequality:

$$\#\mathrm{Fix}\big(f^n\big) \geqslant \sum_{k|n} \big(\#\mathcal{IEOR}\big(f^k\big)\big) \cdot k$$

where $\mathcal{IEOR}(f^k)$ denotes the set of $(\mathcal{I})$ irreducible $(\mathcal{E})$ essential $(\mathcal{O})$ orbits of $(\mathcal{R})$ Reidemeister classes.

On the other hand we have the inequality

$$\mathcal{IEOR}\big(f^k\big) \cdot k \geqslant \mathcal{IER}\big(f^k\big)$$

since each orbit in $\mathcal{IER}(f^k)$ contains at most k elements. Thus we get the inequality

$$\#\mathrm{Fix}\big(f^n\big) \geqslant \sum_{k|n} \big(\#\mathcal{IEOR}\big(f^k\big)\big) \cdot k \geqslant \sum_{k|n} \#\mathcal{IER}\big(f^k\big).$$

The next lemma gives necessary and sufficient condition for the equalities in the above formula.

Lemma 2.2. *Let $f : X \to X$ be a self-map of a compact polyhedron, n a natural number.*

1. *The equality*

$$\#\mathrm{Fix}\big(f^n\big) = \sum_{k|n} \big(\#\mathcal{IEOR}\big(f^k\big)\big) \cdot k$$

holds $\iff$ each orbit of points $a \in OP_k(f)$ (i.e. length of a equals k), represents an orbit of Reidemeister classes $A \in \mathcal{IEOR}(f^k)$ which is represented only by the points in a (for each $k|n$).
In other words natural maps $\mathrm{Fix}(f^k) \to \mathcal{R}(f^k)$ induce bijections

$$\mathcal{O}P_k(f) \to \mathcal{IEOR}\big(f^k\big).$$

2. *The equality*

$$\#\mathrm{Fix}\big(f^n\big) = \sum_{k|n} \#\mathcal{IER}\big(f^k\big)$$

holds $\iff$ each point $x \in P_k(f)$, represents a Reidemeister class $A \in \mathcal{IER}(f^k)$ which is represented only by the point x (for each $k|n$).
In other words natural maps $\mathrm{Fix}(f^k) \to \mathcal{R}(f^k)$ induce bijections

$$P_k(f) \to \mathcal{IER}\big(f^k\big).$$

Proof. 1. Since

$$\#\mathrm{Fix}\big(f^n\big) = \sum_{k|n} \#P_k(f) = \sum_{k|n} \big(\#\mathcal{O}P_k(f)\big) \cdot k$$

and

$$\#OP_k(f) \geqslant \mathcal{IEOR}\big(f^k\big),$$

the equality $\#\mathrm{Fix}(f^n) = \sum_{k|n}(\#\mathcal{IEOR}(f^k)) \cdot k$ holds $\iff$

$$\#\mathcal{O}P_k(f) = \#\mathcal{IEOR}\big(f^k\big)$$

for all $k|n$.

On the other hand we recall that the image of the natural map $OP_k(f) \to \mathcal{OR}(f^k)$ contains $\mathcal{IEOR}(f^k)$, hence the equality exactly means that the above map is a bijection onto $\mathcal{IEOR}(f^k)$, for all $k|n$.

2. We start with the equality

$$\#\mathrm{Fix}\big(f^n\big) = \sum_{k|n} \#P_k(f).$$

Now the equality $\#\mathrm{Fix}(f^n) = \sum_{k|n} \#\mathcal{IER}(f^k)$ holds $\iff$

$$\#P_k(f) = \#\mathcal{IER}\big(f^k\big)$$

for all $k|n$.

On the other hand we recall that last means that the image of the natural map $P_k(f) \to \mathcal{R}(f^k)$ contains $\mathcal{IER}(f^k)$, hence the equality exactly means that the above map is a bijection for all $k|n$. □

Remark 2.3. Let us recall the inequalities

$$\#\mathrm{Fix}\big(f^n\big) \geqslant NJD_n(f) \geqslant NF_n(f) \geqslant \sum_{k|n} \#\mathcal{IER}\big(f^k\big)$$

which are true in general. If moreover equality $\#\mathrm{Fix}(f^n) = \sum_{k|n} \#\mathcal{IER}(f^k)$ takes place then

$$NF_n(f) = NJD_n(f).$$

Thus to show the above equality it is enough to find a smooth map g, homotopic to the given f, and satisfying

each point $x \in P_k(g)$, represents a Reidemeister class $A \in \mathcal{IER}(g^k)$ which is represented only by the point x (for each $k|n$)

(see Lemma 2.2).

3. The least number of periodic points on tori

The Wecken Theorem for periodic points for self-maps of tori is shown in [16]. For a given self-map $f : T^m \to T^m$ and a fixed number $n \in \mathbb{N}$, there is a map g homotopic to f satisfying $\#\mathrm{Fix}(g^n) = \sum_{k|n} NP_k(f)$. The class of maps is not made explicit in [16]. It seems that the author had in mind rather continuous than smooth maps. Nevertheless, it is easy to see that all continuous deformations presented there can be smooth. In fact the proof is done by induction with the respect to the dimension. If $L(f^n) \neq 0$ then a linear map of $\mathbb{R}^m$ induces a map realising the minimal number of periodic points. Otherwise the torus T^m fibres over a torus of a lower dimension, with a torus fibre, and f turns out to be the fibre-map. Since all the deformations performed in [16] come from deformations of the base and of the fibres, they can be made smooth by induction. Especially the deformation from the crucial Lemma 6 in [16] can be made smooth (which is emphasised in the paper [12] cited there). Thus we get equality

$$NF_n(f) = NJD_n(f)$$

for any self-map of a torus.

4. Nilmanifolds and solvmanifolds

A *nilmanifold* is a quotient space of a nilpotent Lie group by its closed subgroup. The simplest examples are tori: the quotient spaces of the commutative Lie groups $T^m = \mathbb{R}^m/\mathbb{Z}^m$. Fadell and Husseini showed that a compact nilmanifold N which is not a torus fibres over another nilmanifold with a torus fibre. Moreover the fibration is the principal fibre bundle, each continuous map $f : N \to N$ is homotopic to a fibre-map and the Nielsen number product formula $N(f) = N(\bar{f}) \cdot N(f_b)$ holds for each such fibre-map [1].

By a *solvmanifold* we mean a quotient space of a solvable Lie group by its closed subgroup. Since a nilpotent group is solvable, each nilmanifold is a solvmanifold. Each compact solvmanifold which is not a nilmanifolds fibres over a torus with a nilmanifold as a fibre (Mostow fibration [4]). Moreover each continuous self-map of a solvmanifold is homotopic to a fibre-map.

Lemma 4.1. *Let $p : E \to N$ be a locally trivial fibre bundle over a nilmanifold, and $f : E \to E$ a fibre-map. Then we have the formula*

$$N(f) = N(f_{b_1}) + \cdots + N(f_{b_s})$$

where $b_1, \ldots, b_s$ is a set of essential representatives, i.e. one point from each essential Nielsen class of $\bar{f} : N \to N$.

Proof. Let us start with the remark that for any self-map of a nilmanifold $f : N \to N$ the equality $|L(f)| = N(f)$ takes place [1].

We consider two cases $L(\bar{f}) = 0$ and $L(\bar{f}) \neq 0$ where $\bar{f} : N \to N$ denotes the map induced by f.

Let $L(\bar{f}) = 0$. Then $N(\bar{f}) = 0$ implies $N(f) = 0$ and the formula holds trivially.

Now we assume that $L(\bar{f}) \neq 0$. By Corollary 4.4.24 in [8] the following two conditions

1. $\pi_2 N = 0$,
2. for any fixed point $x \in \mathrm{Fix}(\bar{f})$ and the induced homomorphism $\bar{f}_\# : \pi_1(N, x) \to \pi_1(N, x)$ the implication
$$\bar{f}_\#(v) = v \quad \Rightarrow \quad v = 1 \in \pi_1(N; x)$$
holds,

imply the formula of the lemma. It remains to show that the above conditions hold for any fibre-map over a nilmanifold satisfying $L(\bar{f}) \neq 0$.

The first condition $\pi_2 N = 0$ holds, since each nilmanifold is a $K(\pi, 1)$ space.

We will have the second condition once we prove that:

For any self-map $h : N \to N$ of a nilmanifold satisfying $L(h) \neq 0$ and a fixed point $x \in \mathrm{Fix}(h)$

$$h_\#(v) = v \quad \text{implies} \quad v = 1 \in \pi_1 N.$$

First we show that the condition holds for self-map of torus $h : T^n \to T^n$ satisfying $L(h) \neq 0$. Now $\pi_1(T^n) = \mathbb{Z}^n$, the homomorphism $h_\# : \mathbb{Z}^n \to \mathbb{Z}^n$ is given by an $(n \times n)$-matrix A and $\det(I - A) = L(h)$. Since $L(h) \neq 0$, $I - A$ in non-singular and 1 is not an eigenvalue of A. Now $v = h_\# v = Av$ implies $v = 1 \in \pi_1 N$.

Now we will show the property for the remaining nilmanifolds. We use the induction with the respect to $n = \dim N$.

For $n = 1, 2$ the nilmanifolds are tori T^1 or T^2. Now we assume that the property holds for all nilmanifolds of dimension $< n$ ($n \geqslant 3$). We consider a self-map $f : N \to N$ of an n-nilmanifold which is not a torus and $L(f) \neq 0$. Then N fibres over another nilmanifold $\bar{N}$ ($\dim \bar{N} < n$) with torus as a fibre. Moreover the map f is homotopic to a fibre-map and we may assume that f itself is a fibre-map. Moreover the equality $L(f) = L(\bar{f}) \cdot L(f')$ holds (here f' is the restriction of f to the fibre over an (arbitrary) fixed point of $\bar{h}$) [1].

Let $x \in \mathrm{Fix}(f)$. We consider the homomorphism $f_\# : \pi_1(N; x) \to \pi_1(N; x)$. Suppose that $f_\#(v) = v$ for a $v \in \pi_1(N; x)$. We have to show that $v = 1$. The fibre-map h induces the map $\bar{f} : \bar{N} \to \bar{N}$ and the homomorphism $\bar{f}_\# : \pi_1(\bar{N}; \bar{x}) \to \pi_1(\bar{N}; \bar{x})$ where $\bar{x} = p(x)$. Now $\bar{f}_\#(\bar{v}) = \bar{v}$ for $\bar{v} = p_\#(v)$, hence the induction assumption implies $\bar{v} = 1 \in \pi_1(\bar{N}; \bar{x})$. Now $v \in \ker p_\#(v) = \mathrm{im}(i_\#)$ where $i : p^{-1}(\bar{x}) \to N$ is the inclusion of the fibre (a torus). Thus $v = i_\#(v')$ for a $v' \in \pi_1(p^{-1}(\bar{x}); x)$. Since $\pi_2 \bar{N} = 0$, $i_\#$ is a monomorphism. Now $f_\#(v) = v$ implies $f'_\#(v') = v'$. Since the fibre $p^{-1}(\bar{x})$ is a torus and $L(f') \neq 0$, $v' = 1$ which implies $v = i_\#(v') = 1$. $\square$

5. Essential reducibility and torality

Heath and Keppelmann introduced some classes of maps and spaces [4,5]. Consider a continuous self-map of a finite polyhedron $f : X \to X$.

We say that f is *essentially reducible* if for Reidemeister classes $A \in \mathcal{R}(f^k)$, $B \in \mathcal{R}(f^l)$ the following implication holds: if A is essential and A reduces to B then B is also essential.

There is a convenient formula for $NF_n(f)$ for an essentially reducible map f.

Lemma 5.1. *([4]) If X is essentially reducible then*

$$NF_n(f) = \sum_{k|n} \#\mathcal{IEOR}\big(f^k\big) \cdot k$$

for any self-map $f : X \to X$ and $n \in \mathbb{N}$.

A map f is called *essentially toral* if

- f is essentially reducible,
- for essential classes $A \in \mathcal{R}(f^k)$, $B, C \in \mathcal{R}(f^l)$ if A reduces to B and C then $B = C$,
- if an orbit $A \in \mathcal{OR}(f^n)$ is essential and irreducible then its length equals n.

Corollary 5.2. *If f is essentially toral then*

$$NF_n(f)=\sum_{k|n}\mathcal{IEOR}\big(f^k\big)\cdot k=\sum_{k|n}\mathcal{IER}\big(f^k\big)$$

for any self-map $f: X\to X$ and $n\in\mathbb{N}$.

Proof. Since X is essentially reducible, $NF_n(f)=\sum_{k|n}\mathcal{IEOR}(f^k)\cdot k$. On the other hand f is toral, hence each essential irreducible orbit in $\mathcal{R}(f^k)$ has length k, which implies $\mathcal{IEOR}(f^k)\cdot k=\mathcal{IER}(f^k)$. □

The space is called essentially reducible (toral) if so are all its self-maps.

Theorem 5.3. *(Proposition 2.4 in [5]) All nilmanifolds and solvmanifolds are essentially toral.*

6. Nilmanifolds and solvmanifolds are smoothly Wecken

Now we will prove the main theorem of the paper.

Theorem 6.1. *Solvmanifolds are smoothly Wecken.*

The proof will be based on the following lemma.

Lemma 6.2. *Let $p: E\to B$ be a locally trivial fibration, where base space and fibre are smooth manifolds and let $f: E\to B$ be a fibre-map satisfying*:

- *the base space and the fibres are smoothly Wecken manifolds of dimension $\geqslant 3$ which are essentially toral or tori (T^1 or T^2),*
- *the Nielsen number "product" formula*

$$N\big(f^k\big)=N\big(f^k_{b_1}\big)+\cdots+N\big(f^k_{b_s}\big)$$

holds for all $k|r$, where $\{b_1,\dots,b_s\}$ is a set of essential representatives (i.e. contains exactly one element from each essential Nielsen class in $N(\bar f^k)$).

Then the map f is fibrewise homotopic to a smooth map g satisfying

$$\#\mathrm{Fix}\big(g^r\big)=NF_r(f).$$

Proof. We may assume that $\bar f$ is smooth and satisfies $\mathrm{Fix}(\bar f^r)=NF_r(\bar f)$, since B is smoothly Wecken by assumption. Now

$$\mathrm{Fix}\big(f^r\big)=\bigcup_{k|r}\ \bigcup_{b\in P_k(\bar f)}\mathrm{Fix}\big(\big(f^k_b\big)^{r/k}\big)$$

is a splitting into mutually disjoint subsets.

Now thanks to Lemma 6.3 and the Homotopy Extension Property we may assume that for each $b\in P_k(\bar f)$ the map $f^k_b: E_b\to E_b$ is smooth and realises the minimal numbers of periodic points for the iterations $(f^k_b)^l$ where $l|(r/k)$.

The theorem will be proved, once we show that the obtained map f satisfies assumption 2 of Lemma 2.2. It remains to show that each point $x\in P_n(f)$ $(n|r)$ belongs to an essential irreducible class consisting of the single point x i.e. $[x]=\{x\}\subset \mathrm{Fix}(f^n)$.

Let us denote $b=p(x)$ and let k be the length of the orbit $b,\bar f(b),\bar f^2(b),\dots$. Since $\bar f$ realises the least number of periodic points, $\{b\}\subset\mathrm{Fix}(\bar f^k)$ is an essential irreducible Nielsen class (B is essentially toral). Now we consider the restriction $(f^k)_b: E_b\to E_b$. Since $x\in P_{n/k}((f^k)_b)$, $\{x\}\subset\mathrm{Fix}((f^k)^{n/k}_b)(=\mathrm{Fix}(f^n_b))$ is an essential irreducible class ($(f_b)^k$ realises the least number of periodic points and the fibre is essentially toral). It remains to show that $\{x\}$ is also the essential irreducible class as the subset of $\mathrm{Fix}(f^n)$.

The class in $\mathrm{Fix}(f^n)$ containing x is essential by the index product formula. Now we show that $[x]=\{x\}\subset\mathrm{Fix}(f^n)$. Suppose that $x'\in[x]\subset\mathrm{Fix}(f^n)$. First we show that $x'\in\mathrm{Fix}(f^n_b)$. In fact px, px' represent the same class in $\mathrm{Fix}(\bar f)^n$, hence $px=px'$. Now $x,x'\in\mathrm{Fix}(f^n_b)=\mathrm{Fix}((f^k_b)^{n/k})$ and we suppose that $x\neq x'$. Since f^k_b realises the least number of periodic points, x and x' represent different essential Nielsen classes in $\mathrm{Fix}(f^n_b)$ and by the "product" formula these classes are contained in different classes in $\mathrm{Fix}(f^n)$. This contradicts to the assumption $x'\in[x]\subset\mathrm{Fix}(f^n)$.

Now we prove that the class $[x]\subset\mathrm{Fix}(f^n)$ is irreducible. Suppose otherwise. Then by the essential reducibility the class $[x]$ contains a point from a shorter orbit, which is impossible, since x is the only point in this class and the length of its orbit is n. □

Lemma 6.3. *Let $M_1, \dots, M_k$ be mutually disjoint smooth manifolds of the same dimension d and let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} M_k \xrightarrow{f_k} M_1$. We denote $f = f_k f_{k-1} \cdots f_1$. If moreover one of the conditions*

1. *$d \geqslant 3$, M_i are smoothly Wecken,*
2. *$d \leqslant 2$ and all M_i are diffeomorphic to a d-torus,*

holds then each map f_i is homotopic to a smooth f_i', $i = 1, \dots, k$, so that $f' = f_k' f_{k-1}' \cdots f_1'$ realises the least number of periodic points i.e.

$$NF_n(f) = \#\mathrm{Fix}\big(\big(f'\big)^n\big).$$

Proof. Let $M_i \approx S^1$ for $i = 0, \dots, k-1$. We may identify $M_i = S^1$ by diffeomorphisms which we may choose so that $f_i : M_i \to M_{i+1}$ has $\deg(f_i) \geqslant 0$ for $i = 1, \dots, k-1$ and $\mathrm{sign}(\deg(f_k)) = \mathrm{sign}(\deg(f))$. We may also assume that $f_i(z) = z^{d_i}$ where $d_i = \deg(f_i)$. Let us denote $d = \deg(f) = d_1 \cdots d_k$.

If $d = \pm 1$ then $d_1 = \cdots = d_{k-1} = +1$, hence by the above notations $f_1 = \cdots = f_{k-1} = \mathrm{id}_{S^1}$. Now we may deform f_k to a map f' minimising the set $NF_n(f)$. Then so also does the composition $f_K f_{k-1} \cdots f_1 = f'$.

If $d \neq \pm 1$ then the composition $f = f_k f_{k-1} \cdots f_1$ realises the least number of periodic points, for each multiplicity.

Now we consider the case $M_i \approx T^2$. We will denote M_i by T_i. We may assume that $f_i : T_i \to T_{i+1}$ are induced by linear maps $\tilde{f}_i : \mathbb{R}^2 \to \mathbb{R}^2$. The case $\dim(f(\mathbb{R}^2)) = 0$ is trivial, hence we assume that $\dim(f(\mathbb{R}^2)) = 1$. This means that $f_i(T_i) = S^1$ for an $i = 1, \dots, k$. Let us assume that $f_k(T_k)$ is a circle. We denote $S_1 = f_k(T_k)$, $S_2 = f_1(T_1), \dots, S_k = f_1(T_{k-1})$. Let $f_i' : S_i \to S_{i+1}$ denote the restriction of f_i. Since $f_k \cdots f_1(T_1) = S_1$, the commutativity implies $\mathrm{Fix}((f_k' \cdots f_1')^n) = \mathrm{Fix}((f_k \cdots f_1)^n)$ and $NF_n(f_k' \cdots f_1') = NF_n(f_k \cdots f_1)$. On the other hand, by the first part, there exist homotopies f_{it}' from $f_{i0}' = f_i'$ to maps f_{i1}' satisfying $\#\mathrm{Fix}((f_{k1}' \cdots f_{11}')^r) = NF_r(f_{k1}' \cdots f_{11}')$. Now any extension of these homotopies to $f_{it} : T_i \to T_{i+1}$ with $f_{kt}(T_k) \subset S_1$, satisfies

$$\#\mathrm{Fix}\big((f_{k1} \cdots f_{11})^r\big) = \#\mathrm{Fix}\big(\big(f_{k1}' \cdots f_{11}'\big)^r\big) = NF_r\big(f_{k1}' \cdots f_{11}'\big) = NF_r(f_{k1} \cdots f_{11}).$$

If $f_k(T_k)$ is not a circle but so is $f_i(T_i)$ then we apply the above to the composition $f_{i-1} \cdots f_1 f_n \cdots f_i$.

Now we assume that no $\tilde{f}_i$ is singular. Then all $f_i : T_i \to T_{i+1}$ are finite coverings. Let $h_t : T_1 \to T_1$ be a deformation starting from $h_0 = f_k \cdots f_1$ to a map h_1 satisfying $\#\mathrm{Fix}(h_1^n) = NF_n(f_k \cdots f_1)$. Then we consider the diagram

$$\begin{tikzcd} & T_2 \arrow[d, "f_n \cdots f_2"] \\ T_1 \arrow[ur, "\tilde{h}_t"] \arrow[r, "h_t"'] & T_1. \end{tikzcd}$$

Then the homotopy h_t admits a lift $\tilde{h}_t : T_1 \to T_2$ satisfying $h_t = f_k \cdots f_2 \tilde{h}_t$. In particular $f_k \cdots f_2 \tilde{h}_1 = h_1$ realises the least number of periodic points.

Now we assume that $\dim M_i = d \geqslant 3$. Now the theorem follows from the remark that the Wecken-type Theorem for periodic points (Theorem 2.5 in [7]: each self-map $f : M \to M$ of a compact manifold of dimension $\geqslant 3$ is homotopic to a continuous map g satisfying $\#\mathrm{Fix}(g^n) = NF_n(f)$) also holds without the assumption that M is connected (but all components have the same dimension $\geqslant 3$). In fact there are two kinds of homotopies in [6] and [7]: local ones (in Euclidean neighbourhoods) and a large homotopy deforming the image $f^k\omega$ of a path establishing the Nielsen relation to a path close to ω. The arguments the deformation is possible rely only on transversality and the assumption that the manifold is connected is inessential. $\square$

Proof of Theorem 6.1. First we prove the theorem for nilmanifolds. Tori are smoothly Wecken by Section 3. Now we use the induction with the respect to the dimension of a nilmanifold. We consider a nilmanifold N which is not a torus. Then N fibres over another nilmanifold and we may assume that the base space and the fibre are smoothly Wecken nilmanifolds. Moreover we may assume that f is fibrewise. Now we may apply Lemma 6.2, since the "product" formula holds by Lemma 4.1.

Now we consider a self-map of a solvmanifold $f : S \to S$. We recall that S fibres over a torus with a nilmanifold as a fibre (Mostow fibration). Moreover f is homotopic to a fibre-map, hence we may assume that f is fibrewise and we may apply Lemma 6.2. $\square$

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